

# Krull dimensions of rings of holomorphic functions

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**ABSTRACT.** We prove that the Krull dimension of the ring of holomorphic functions of a connected complex manifold is at least the cardinality of continuum iff it is  $> 0$ .

Let  $R$  be a commutative ring. Recall that the *Krull dimension*  $\dim(R)$  of  $R$  is the supremum of cardinalities lengths of chains of distinct proper prime ideals in  $R$ . Our main result is:

**THEOREM 1.** Let  $M$  be a connected complex manifold and  $H(M)$  be the ring of holomorphic functions on  $M$ . Then the Krull dimension of  $H(M)$  either equals 0 (iff  $H(M) = \mathbb{C}$ ) or is infinite, iff  $M$  admits a nonconstant holomorphic function  $M \rightarrow \mathbb{C}$ . More precisely, unless  $H(M) = \mathbb{C}$ ,  $\dim H(M) \geq \mathfrak{c}$ , i.e., the ring  $H(M)$  contains a chain of distinct prime ideals whose length has cardinality of continuum.

Our proof of this theorem mostly follows the lines of the proof by Sasane [S], who proved that for each nonempty domain  $M \subset \mathbb{C}$  the Krull dimension of  $H(M)$  is infinite (he did not prove that  $\dim H(M) \geq \mathfrak{c}$ ).

**REMARK 2.** We note that Henriksen [H] was the first to prove that the Krull dimension of the ring of entire functions on  $\mathbb{C}$  has cardinality at least continuum.

In our proof we will use the Axiom of Choice in two ways: (a) to establish existence of certain maximal ideals and (b) to get existence of a nonprincipal ultrafilter  $\omega$  on  $\mathbb{N}$  and, hence of the ordered field  ${}^*\mathbb{R}$  of *nonstandard real* (or, *surreal*) numbers. The field  ${}^*\mathbb{R}$  contains  ${}^*\mathbb{N}$ , the *nonstandard natural* (or *surnatural*) numbers.

The field  ${}^*\mathbb{R}$  is a certain quotient of the countable direct product  $\prod_{k \in \mathbb{N}} \mathbb{R}$ ; we will denote the equivalence class (in  ${}^*\mathbb{R}$ ) of a sequence  $(x_k)$  in  $\mathbb{R}$  by  $[x_k]$ . Accordingly,  ${}^*\mathbb{N}$  consists of equivalence classes  $[n_k]$  of sequences of natural numbers. Roughly speaking, we will use  ${}^*\mathbb{N}$  and certain order relation on it to compare rates of growth of sequences of natural numbers.

**DEFINITION 3.** A commutative unital ring  $R$  is *ample* if there exists a sequence of valuations  $\nu_k$  on  $R$  such that for each  $\beta \in {}^*\mathbb{N}$ , there  $a = a_\beta \in R$  with the property

$$(1) \quad [\nu_k(a)] = \beta.$$

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The main technical result of this paper is:

**THEOREM 4.** For each ample ring  $R$ ,  $\dim(R) \geq \mathfrak{c}$ . In particular,  $R$  has infinite Krull dimension.

This theorem and its proof are inspired by Theorem 2.2 of [S], although some parts of the proof resemble the ones of [H].

We will verify, furthermore, that whenever  $M$  is a connected complex manifold which has a nonconstant holomorphic function, the ring  $H(M)$  is ample. This, combined with Theorem 4, will immediately imply Theorem 1.

**REMARK 5.** 1. We refer the reader to Section 5.3 of [Cla] for further discussion of algebraic properties of rings of holomorphic functions.

2. Theorem 1 shows that for every Stein manifold  $M$  (of positive dimension), the ring  $H(M)$  has infinite Krull dimension. In particular, this applies to any noncompact connected Riemann surfaces (since every such surface is Stein, [BS]).

3. Noncompact connected complex manifolds  $M$  of dimension  $> 1$  can have  $H(M) = \mathbb{C}$ ; for instance, take  $M$  to be the complement to a finite subset in a compact connected complex manifold (of dimension  $> 1$ ).

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## 1. Surreal numbers

We refer the reader to [Go] for a detailed treatment of surreal numbers, below is a brief introduction. A nonprincipal ultrafilter on  $\mathbb{N}$  can be regarded as a finitely-additive probability measure on  $\mathbb{N}$  which vanishes on each finite subset and takes the value 0 or 1 on each subset of  $\mathbb{N}$ . The existence of nonprincipal ultrafilters (the *ultrafilter lemma*) follows from the Axiom of Choice. Subsets of full measure are called  $\omega$ -large. Using  $\omega$  one defines the following equivalence relation on the product

$$\prod_{k \in \mathbb{N}} \mathbb{R}.$$

Two sequences  $(x_k)$  and  $(y_k)$  are equivalent if  $x_k = y_k$  for an  $\omega$ -all  $k$ , i.e. the set

$$\{k : x_k = y_k\}$$

is  $\omega$ -large. The quotient by this equivalence relation, denoted

$${}^*\mathbb{R} = \prod_{k \in \mathbb{N}} \mathbb{R} / \omega,$$

is the set of surreal numbers. Let  $[x_k]$  be the equivalence class of the sequence  $(x_k)$ .

The binary operations on sequences of real numbers project to binary operations on  ${}^*\mathbb{R}$  making  ${}^*\mathbb{R}$  a field. The total order  $\leq$  on  ${}^*\mathbb{R}$  is defined by  $[x_k] \leq [y_k]$  iff  $x_k \leq y_k$  for an  $\omega$ -all  $k \in \mathbb{N}$ . With this order,  ${}^*\mathbb{R}$  becomes an ordered field.

The set of real numbers embeds into  ${}^*\mathbb{R}$  as the set of equivalence classes of constant sequences; the image of a real number  $x$  under this embedding is still denoted  $x$ . We set  ${}^*\mathbb{R}_+ := \{\alpha \in {}^*\mathbb{R} : \alpha > 0\}$ .

The projection of

$$\prod_{k \in \mathbb{N}} \mathbb{N} \subset \prod_{k \in \mathbb{N}} \mathbb{R}$$

to  ${}^*\mathbb{R}$  is denoted  ${}^*\mathbb{N}$ , this is the set of *surnatural numbers*. We define a further equivalence relation  $\sim_u$  on  ${}^*\mathbb{R}$  by:

$$\alpha \sim_u \beta$$

if there exist positive real numbers  $a, b$  such that

$$a\alpha \leq \beta \leq b\alpha.$$

The equivalence class  $(\alpha)$  of  $\alpha \in {}^*\mathbb{R}$  (for this equivalence relation) is a multiplicative analogue of the *galaxy*  $gal(\alpha)$  of  $\alpha$ , see [Go]:

DEFINITION 6. The *galaxy*  $gal(\alpha)$  of a surreal number  $\alpha \in {}^*\mathbb{R}$  is the union

$$\bigcup_{n \in \mathbb{N}} [\alpha - n, \alpha + n] \subset {}^*\mathbb{R}.$$

In other words,  $\beta \in gal(\alpha)$  iff there exist a real number  $a$  such that  $\alpha - a \leq \beta \leq \alpha + a$ .

The next lemma is immediate:

LEMMA 7. For  $\alpha \in {}^*\mathbb{R}_+$ , the equivalence class  $(\alpha)$  of  $\alpha$  equals  $\exp(gal(\log(\alpha)))$ .

We let  ${}^u\mathbb{R}$  denote the quotient  ${}^*\mathbb{R} / \sim_u$  and  ${}^u\mathbb{N}$  the projection of  ${}^*\mathbb{N}$  to  ${}^u\mathbb{R}$ . Define the total order  $\gg$  on  ${}^u\mathbb{R}$  by

$$(\beta) \gg (\alpha)$$

if for every real number  $c$ ,  $c\alpha < \beta$ . By abusing the notation, we will simply say that  $\beta \gg \alpha$ , with  $\alpha, \beta \in {}^*\mathbb{R}$ .

For the reader who prefers to think in terms of sequences of (positive) real numbers, the relation  $(\beta) \gg (\alpha)$  is an analogue of the relation

$$(a_n) = o((b_n)), \quad n \rightarrow \infty.$$

REMARK 8. The equivalence relation  $\sim_u$  and the order  $\gg$  are similar to the ones used by Henricksen in [H].

PROPOSITION 9. The set  ${}^u\mathbb{N}$  has the cardinality of continuum.

*Proof.* Note first, that  ${}^*\mathbb{R}$  has cardinality of continuum, hence, the cardinality of  ${}^u\mathbb{N}$  is at most  $\mathfrak{c}$ . The proof of the proposition then reduces to two lemmata.

LEMMA 10. The set  $gal({}^*\mathbb{R}_+)$  of galaxies  $\{gal(\alpha) : \alpha \in {}^*\mathbb{R}_+\}$  has the cardinality of continuum.

*Proof.* For each  $\alpha = [a_k] \in {}^*\mathbb{R}_+$ , the galaxy  $gal(\alpha)$  contains the surnatural number  $[\alpha] = [b_k]$ , where  $b_k = [a_k]$ . For each surnatural number  $\beta \in {}^*\mathbb{N}$ , and natural number  $n \in \mathbb{N}$ , the intersection

$$[\beta - n, \beta + n] \cap {}^*\mathbb{N}$$

is finite, equal  $\{\beta - n, \dots, \beta + n\}$ . Therefore,  $gal(\beta) \cap {}^*\mathbb{N} = \{\beta\} + \mathbb{Z}$ . It follows that the map

$${}^*\mathbb{N} \rightarrow gal({}^*\mathbb{R}_+), \quad \beta \mapsto gal(\beta)$$

is a bijection modulo  $\mathbb{Z}$ . Lastly, the set of surnatural numbers  ${}^*\mathbb{N}$  has the cardinality of continuum.  $\square$

LEMMA 11. The map  $\lambda : {}^*\mathbb{N} \rightarrow \text{gal}({}^*\mathbb{R}_+)$ ,  $\lambda : \beta \mapsto \text{gal}(\log(n))$ , is surjective.

*Proof.* For each  $\alpha \in {}^*\mathbb{R}_+$  let  $\beta = \lceil \exp(\alpha) \rceil \in {}^*\mathbb{N}$ . Since  $\log(x+1) - \log(x) \leq 1$  for  $x \geq 1$ , we have that

$$\log(\beta) \in \text{gal}(\alpha). \quad \square$$

Now, we can finish the proof of the proposition. The map  $\lambda : {}^*\mathbb{N} \rightarrow \text{gal}({}^*\mathbb{R}_+)$  descends to a map  $\mu : {}^u\mathbb{N} \rightarrow \text{gal}({}^*\mathbb{R}_+)$ . According to Lemma 11, the map  $\mu$  is surjective. By Lemma 10 the set  $\text{gal}({}^*\mathbb{R}_+)$  has the cardinality of continuum.  $\square$

We will prove Theorem 4 in the next section by showing that for each ample ring  $R$ , the ordered set  $({}^u\mathbb{N}, \gg)$  embeds into the poset of prime ideals in  $R$  reversing the order:

$$(\beta) \gg (\alpha) \Rightarrow P_\beta \subsetneq P_\alpha$$

for certain prime ideals  $P_\gamma \subset R$  determines by  $(\gamma) \in {}^u\mathbb{N}$ . Proposition 9 will then imply that the Krull dimension of  $R$  is at least  $\mathfrak{c}$ .

## 2. Krull dimension of ample rings

Recall that a *valuation* on a unital ring  $R$  is a map  $\nu : R \rightarrow \mathbb{R}_+ \cup \{\infty\}$  such that:

1.  $\nu(a+b) \geq \min(\nu(a), \nu(b))$ ,
2.  $\nu(ab) = \nu(a) + \nu(b)$ .
3.  $\nu(a) = \infty \iff a = 0$ .
4.  $\nu(1) = 0$ .

For the following lemma, see Theorem 10.2.6 in [Coh] (see also Proposition 4.8 of [Cla] or Theorem 1 in [K]).

LEMMA 12. Let  $I$  be an ideal in a commutative ring  $A$  and  $M \subset A \setminus I$  be a subset closed under multiplication. Then there exists an ideal  $J \subset A$  containing  $I$  and disjoint from  $M$ , so that  $J$  is maximal with respect to this property. Furthermore,  $J$  is a prime ideal in  $A$ .

Let  $R$  be an ample ring and  $\nu_k$  the corresponding sequence of valuations on  $R$ . For each  $\beta \in {}^*\mathbb{N}$  we define

$$I_\beta := \{a \in R \mid [\nu_k(a)] \gg [\beta]\} \subset R.$$

LEMMA 13. Each  $I_\alpha$  is an ideal in  $R$ .

*Proof.* We will check that  $I_\alpha$  is additive since it is clearly closed under multiplication by elements of  $R$ . Take  $p', p'' \in I_\alpha$ ,

$$[\nu_k(p')] \gg \alpha, [\nu_k(p'')] \gg \alpha.$$

By the definition of a valuation,

$$n_k := \nu_k(p' + p'') \geq \min(\nu_k(p'), \nu_k(p'')),$$

for each  $k \in \mathbb{N}$ . For  $m \in \mathbb{N}$ , define the  $\omega$ -large sets

$$A' = \{k : \nu_k(p') \geq m\alpha\}, \quad A'' = \{k : \nu_k(p'') \geq m\alpha\}.$$

Therefore, their intersection  $A = A' \cap A''$  is  $\omega$ -large as well, which implies that

$$\forall m \in \mathbb{N}, [n_k] \geq m\alpha \Rightarrow [n_k] \gg \alpha. \quad \square$$

Then for each  $\gamma \gg \beta$ , the element  $a_\gamma$  as in Definition 3, belongs to  $I_\beta$ . It follows that  $I_\beta \neq 0$  for every  $\beta$ . Define the subsets

$$M_\beta := \{a \in R \mid \exists n \in \mathbb{N}, [\nu_k(a)] \leq n\beta\} \subset R;$$

each  $M_\beta$  is closed under the multiplication. It is immediate that whenever  $\alpha \leq \beta$ , we have the inclusions

$$I_\beta \subset I_\alpha, \quad M_\alpha \subset M_\beta.$$

It is also clear that  $I_\beta \cap M_\beta = \emptyset$ . At the same time, for each  $\beta \gg \alpha$ ,

$$a_\beta \in I_\alpha \cap M_\beta.$$

For each  $\alpha$  we let  $\mathcal{J}_\alpha$  denote the set of ideals  $P \subset R$  such that

$$I_\alpha \subset P, P \cap M_\alpha = \emptyset.$$

By Lemma 12, every maximal element  $P \in \mathcal{J}_\alpha$  is a prime ideal.

LEMMA 14. Every  $\mathcal{J}_\alpha$  contains unique maximal element, which we will denote  $P_\alpha$  in what follows.

*Proof.* Suppose that  $P', P''$  are two maximal elements of  $\mathcal{J}_\alpha$ . We define the ideal  $P = P' + P''$ . Clearly,  $P$  contains  $I_\alpha$ . To prove that  $P$  is disjoint from  $M_\alpha$ , take  $p' \in P', p'' \in P''$ , since  $p' \notin M_\alpha, p'' \notin M_\alpha$ . Then the same proof as in Lemma 13 shows that  $[\nu_k(p' + p'')] \gg \alpha$  which means that  $p' + p'' \notin M_\alpha$ . Thus,  $P \in \mathcal{J}_\alpha$  and, in view of maximality of  $P', P''$ , we obtain

$$P' = P = P''. \quad \square$$

For each  $\beta \gg \alpha$  we define the ideal  $Q_{\alpha\beta} := I_\alpha + P_\beta$ .

LEMMA 15.  $Q_{\alpha\beta} \cap M_\alpha = \emptyset$ .

*Proof.* The proof is similar to the one of the previous lemma. Let  $q = c + p$ ,  $c \in I_\alpha, p \in P_\beta$ . Since  $p \notin M_\beta, p \notin M_\alpha$  as well. Therefore,

$$[\nu_k(p)] \gg \alpha.$$

Since  $c \in I_\alpha$ ,

$$[\nu_k(c)] \gg \alpha.$$

Hence,

$$[\nu_k(c + p)] \gg \alpha$$

as well. Thus,  $q \notin M_\alpha$ .  $\square$

COROLLARY 16.  $Q_{\alpha\beta} \in \mathcal{J}_\alpha$ . In particular,  $Q_\alpha \subset P_\alpha$ .

*Proof.* It suffices to note that  $I_\alpha \subset Q_{\alpha\beta}$  according to the definition of  $Q_{\alpha\beta}$ .  $\square$

LEMMA 17. The inequality  $\beta \gg \alpha$  implies  $P_\beta \subset P_\alpha$  and this inclusion is proper.

*Proof.* By the definition of  $Q_{\alpha\beta}$  and Corollary 16, we have the inclusions

$$P_\beta \subset Q_\alpha \subset P_\alpha.$$

We now claim that  $P_\beta \neq Q_{\alpha\beta} = I_\alpha + P_\beta$ . Recall that  $a_\alpha \in I_\alpha \subset Q_{\alpha\beta}$  and  $a_\alpha \in M_\beta$ , while  $M_\beta \cap P_\beta = \emptyset$ . Thus,  $a_\alpha \in Q_{\alpha\beta} \setminus P_\beta$ .  $\square$

According to Proposition 9, the set  ${}^*\mathbb{N}$  of surnatural numbers contains a subset  $S$  of cardinality continuum such that for all  $\alpha < \beta$  in  $S$ , we have  $\beta \gg \alpha$ . The map

$$\alpha \mapsto P_\alpha$$

sends each  $\alpha \in S$  to a prime ideal in  $R$ ;  $\alpha < \beta$  implies that  $P_\beta \subsetneq P_\alpha$ .

We conclude that the ring  $R$  contains the (descending) chain of distinct prime ideals  $P_\alpha, \alpha \in S$ ; the length of this chain has the cardinality of continuum. In particular,  $\dim(R) \geq \mathfrak{c}$ . Theorem 4 follows.  $\square$

### 3. Ampleness of rings of holomorphic functions

We will need the following classical result, see e.g. [Con, Ch. VII, Theorem 5.15]:

**THEOREM 18.** Let  $D \subset \mathbb{C}$  be a domain, and let  $c_k \in D$  be a sequence which does not accumulate anywhere in  $D$  and let  $m_k$  be a sequence of natural numbers. Then there exists a holomorphic function  $g$  in  $D$  which has zeroes only at the points  $c_k$  and such that  $m_k$  is the order of zero of  $g$  at  $c_k$ ,  $k \in \mathbb{N}$ .

**COROLLARY 19.** If  $M$  is a connected complex manifold which admits a non-constant holomorphic function  $h : M \rightarrow \mathbb{C}$ , then the ring  $H(M)$  is ample.

*Proof.* We let  $D$  denote the image of  $h$ . Pick a sequence  $c_k \in D$  which converges to a point in  $\hat{\mathbb{C}} \setminus D$  and which consists of regular values of  $h$ . (Here  $\hat{\mathbb{C}}$  is the Riemann sphere.) For each  $c_k$  the preimage  $C_k := h^{-1}(c_k)$  is a complex submanifold in  $M$ ; in each  $C_k$  pick a point  $b_k$ . Define valuations

$$\nu_k : H(M) \rightarrow \mathbb{Z}_+ \cup \{\infty\}$$

by  $\nu_k(f) := \text{ord}_{b_k}(f)$ , the total order of  $f$  at  $b_k$ , cf. [Gu, Chapter C, Definition 1].

Now, given  $\beta \in {}^*\mathbb{N}$ ,  $\beta = [m_k]$ , we let  $g = g_\beta$  denote a holomorphic function on  $D$  as in Theorem 18. Define  $a = a_\beta := g \circ h \in H(M)$ . Then  $\nu_k(a) = m_k$ , which implies that the ring  $H(M)$  is ample.  $\square$

Ampleness of  $H(M)$  together with Theorem 4 imply Theorem 1.

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